Eur. Phys. J. C 20, 585–591 (2001)
Digital Object Identifier (DOI) 10.1007/s100520100657

THE EUROPEAN PHYSICAL JOURNAL C

Path integral for spinning particle in the plane wave field: Global and local projections

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Received: 19 February 2001 /
Published online: 28 May 2001 –

Abstract. The Green function related to the problem of a Dirac particle interacting with a plane wave is calculated via the path integral formalism proposed recently by Alexandrou et al. according to the two so-called global and local projections. With the help of the incorporation of two simple identities, it is shown that the contribution to the calculation of the integrals comes essentially from classical solutions projected along the direction of wave propagation.

1 Green function calculations

Recently, another formulation using path integral [1] (which does not introduce the matrix γ^5 as opposed to the previously existing one [2]) was proposed to describe the dynamics of Dirac particles. Thus, the Green function of a system can be obtained in two ways.

First, we have the so-called global projection

$$G_{\mathrm{g}}(x,y)=\frac{-\mathrm{i}}{2k_0}\left(\mathrm{i}\widehat{\partial}-g\widehat{A}(x)+m\right)\widetilde{G}_{\mathrm{g}}(x,y), \qquad (1\mathrm{a})$$

where

$$\widetilde{G}_{\mathrm{g}}(x,y)=\exp\left(\gamma\cdot\frac{\partial}{\partial\Gamma}\right)\int_0^\infty \mathrm{d}T\int DxD\pi\mathcal{D}\zeta \times\exp\left\{\mathrm{i}\int_0^T\left[\frac{1}{2k_0}\left(\pi^2-m^2\right)-\frac{k_0}{2}\dot{x}^2- g\dot{x}\cdot A(x)+\mathrm{i}\zeta\cdot\dot{\zeta}-\frac{\mathrm{i}g}{k_0}F_{\mu\nu}(x)\zeta^\mu\zeta^\nu\right]\mathrm{d}\tau+ \zeta(0)\cdot\zeta(T)\right\}\Bigg|_{\Gamma=0}, \qquad (1\mathrm{a'})$$

$$\mathcal{D}\zeta=\left[\int D\zeta\exp\left(\zeta(0)\zeta(T)-\int_0^T\mathrm{d}\tau\zeta\,\dot{\zeta}\right)\right]^{-1}D\zeta,$$

where the parameter k_0 plays the role of a mass, $\widehat{a}=\gamma\cdot a$ and the operator $\left(\mathrm{i}\widehat{\partial}-g\widehat{A}(x)+m\right)$ is used to eliminate the superfluous states.

The second possibility is by the so-called local projection

$$G_{\mathrm{l}}(x,y)=\exp\left(\gamma\cdot\frac{\partial}{\partial\Gamma}\right)\int_0^\infty\mathrm{d}T\int\mathrm{d}\chi \times\exp\left[\frac{-\mathrm{i}}{2k_0}\left(m^2T+m\chi\right)\right]\int DxD\pi\mathcal{D}\zeta \times\exp\left\{\mathrm{i}\int_0^T\left[\frac{\pi^2}{2k_0}-\frac{k_0}{2}\dot{x}^2-g\dot{x}\cdot A(x) +\mathrm{i}\zeta\cdot\dot{\zeta}-\frac{\mathrm{i}g}{k_0}F_{\mu\nu}(x)\zeta^\mu\zeta^\nu+\frac{1}{T}\dot{x}\cdot\zeta\chi\right]\mathrm{d}\tau +\zeta(0)\cdot\zeta(T)\right\}\Bigg|_{\Gamma=0}, \qquad (1\mathrm{b})$$

where $\Gamma^\mu, \mu=\overline{0,4}$ and χ are odd (Grassmann) variables anticommuting with the γ-matrices.

The boundary conditions for the x-space path integral are

$$x_\mu(0)=y_\mu,\quad x_\mu(T)=x_\mu,$$

and the antiperiodic boundary condition for the Grassmann path integral is

$$\zeta_\mu(0)+\zeta_\mu(T)=\Gamma_\mu,$$

where $\zeta_\mu(\tau)$, $\mu=\overline{0,4}$ are odd trajectories, anticommuting with the γ-matrices.

The object of this paper is to see how we can obtain from this recent formalism the solution for the simple case of a Dirac particle interacting with a plane wave field by determining exactly the corresponding Green function in the two representations, global and local. The configuration of the field is characterized by the following features:

(1) the four-vector potential is an arbitrary function of $\varphi=k\cdot x$ with $k^2=0$, where k is the plane wave propagation four-vector,

(2) the external field $A_\mu(\varphi)$ satisfies the transversality condition (Lorentz gauge) $k\cdot A=0$. The scalar product of four-vectors, denoted by a dot, stands for $a\cdot b=a_\mu b^\mu$.

By introducing only two identities, one profiting from the feature of the plane wave (a feature already dealt with in [3–5]), another quite similar but taking into account the form of the spinning interaction, we are going to show the importance of the classical equations of motion for such a problem while determining the computations of the expressions (1a') and (1b).

As the plane wave field is only a function of the product $k\cdot x$, let us first assume that the variable $k\cdot x$ is independent of the four-position x by introducing the following identity:

$$\int \mathrm{d}\varphi_a \mathrm{d}\varphi_b \delta\left(\varphi_a - k\cdot y\right) \times \int D\varphi Dp_\varphi \exp\left[\mathrm{i}\int_0^T p_\varphi\left(\dot{\varphi} - k\cdot\dot{x}\right)\mathrm{d}\tau\right] = 1, \tag{2}$$

expressed with the phase space variables (φ, p_φ).

By inserting respectively this identity in (1a') and (1b), we get for the expressions of the Green functions the following results, respectively:

$$\begin{aligned}\widetilde{G}_{\mathrm{g}}(x,y) = {} & \exp\left(\gamma\cdot\frac{\partial}{\partial\Gamma}\right)\int_0^\infty \mathrm{d}T \int \mathrm{d}\varphi_a \mathrm{d}\varphi_b \delta\left(\varphi_a - k\cdot y\right) \\ & \times\int Dx D\pi D\varphi Dp_\varphi \mathcal{D}\zeta \exp\Bigg\{\mathrm{i}\int_0^T\Bigg[\frac{-k_0}{2}\dot{x}^2 - \\ & \dot{x}\cdot\left[gA\left(\varphi\right) + kp_\varphi\right] + \frac{1}{2k_0}\left(\pi^2 - m^2\right) \\ & +\mathrm{i}\zeta\cdot\dot{\zeta} - \frac{\mathrm{i}g}{k_0}F_{\mu\nu}(\varphi)\zeta^\mu\zeta^\nu + p_\varphi\dot{\varphi}\Bigg]\mathrm{d}\tau \\ & +\zeta\left(0\right)\cdot\zeta\left(T\right)\Bigg\}\Bigg|_{\Gamma=0},\end{aligned} \tag{2a}$$

$$\begin{aligned}G_{\mathrm{l}}\left(x,y\right) = {} & \exp\left(\gamma\cdot\frac{\partial}{\partial\Gamma}\right)\int_0^\infty \mathrm{d}T\int \mathrm{d}\chi \\ & \times\exp\left[\frac{-\mathrm{i}}{2k_0}\left(m^2T + m\chi\right)\right] \\ & \times\int \mathrm{d}\varphi_a \mathrm{d}\varphi_b \delta\left(\varphi_a - k\cdot y\right) \\ & \times\int Dx D\pi D\varphi Dp_\varphi \mathcal{D}\zeta \\ & \times\exp\Bigg\{\mathrm{i}\int_0^T\Bigg[\frac{-k_0}{2}\dot{x}^2 + \\ & \dot{x}\cdot\left(\frac{1}{T}\zeta\chi - gA(\varphi) - kp_\varphi\right) \\ & +\frac{\pi^2}{2k_0} - \frac{\mathrm{i}g}{k_0}F_{\mu\nu}(\varphi)\zeta^\mu\zeta^\nu + \mathrm{i}\zeta\cdot\dot{\zeta} + p_\varphi\dot{\varphi}\Bigg]\mathrm{d}\tau \\ & +\zeta(0)\zeta(T)\Bigg\}\Bigg|_{\Gamma=0}.\end{aligned} \tag{2b}$$

To be able to integrate over the x variables let us linearize, as usually is done, the kinetic term by introducing the momentum variables p. The integrations give a delta functional $\delta\left(\dot{p}\right)$ which implies that the momentum p is conserved during the evolution ($p = \mathrm{cons}$).

Accordingly the results of the Green functions in the two projections are written

$$\begin{aligned}\widetilde{G}_{\mathrm{g}}(x,y) = {} & \exp\left(\gamma\cdot\frac{\partial}{\partial\Gamma}\right)\int_0^\infty \mathrm{d}T\int\frac{\mathrm{d}^4p}{(2\pi)^4}\int \mathrm{d}\varphi_a \mathrm{d}\varphi_b \\ & \times\delta\left(\varphi_a - k\cdot y\right)\int D\varphi Dp_\varphi \mathcal{D}\zeta \exp\Bigg\{\mathrm{i}p\cdot(x-y) \\ & +\frac{\mathrm{i}T}{2k_0}(p^2 - m^2) + \mathrm{i}\int_0^T\Bigg[\frac{g}{k_0}\left(A\cdot p + \frac{g}{2}A^2\right) \\ & +\mathrm{i}\zeta\cdot\dot{\zeta} + p_\varphi\left(\dot{\varphi} + \frac{p\cdot k}{k_0}\right) - \frac{\mathrm{i}g}{k_0}F_{\mu\nu}(\varphi)\zeta^\mu\zeta^\nu\Bigg]\mathrm{d}\tau \\ & +\zeta(0)\cdot\zeta\left(T\right)\Bigg\}\Bigg|_{\Gamma=0},\end{aligned} \tag{3a}$$

$$\begin{aligned}G_{\mathrm{l}}(x,y) = {} & \exp\left(\gamma\cdot\frac{\partial}{\partial\Gamma}\right)\int_0^\infty \mathrm{d}T\int\frac{\mathrm{d}^4p}{(2\pi)^4}\int \mathrm{d}\chi \\ & \times\exp\left[\frac{-\mathrm{i}m}{2k_0}\chi\right]\int \mathrm{d}\varphi_a \mathrm{d}\varphi_b \delta\left(\varphi_a - k\cdot y\right) \\ & \times\int D\varphi Dp_\varphi\int\mathcal{D}\zeta \exp\Bigg\{\mathrm{i}p\cdot(x-y) \\ & +\frac{\mathrm{i}T}{2k_0}(p^2 - m^2) + \mathrm{i}\int_0^T\Bigg[\frac{g}{k_0}\left(A\cdot p + \frac{g}{2}A^2\right) \\ & +p_\varphi\left(\dot{\varphi} + \frac{p\cdot k}{k_0}\right) - \frac{\mathrm{i}g}{k_0}F_{\mu\nu}(\varphi)\zeta^\mu\zeta^\nu + \mathrm{i}\zeta\cdot\dot{\zeta} \\ & -\frac{1}{k_0T}\left[(gA + kp_\varphi + p)\cdot\zeta\right]\chi\Bigg]\mathrm{d}\tau \\ & +\zeta\left(0\right)\cdot\zeta(T)\Bigg\}\Bigg|_{\Gamma=0}.\end{aligned} \tag{3b}$$

The introduction of this constraint has allowed us, therefore, to bring the study of motion from four-dimensional space to a one-dimensional space described by the variable φ. Furthermore, noting that the coupling term of the spin variables with the electromagnetic field can be written as

$$F_{\mu\nu}(\varphi)\zeta^\mu\zeta^\nu = 2\left(k\cdot\zeta\right)\left(A'\cdot\zeta\right), \tag{4}$$

where the prime indicates a derivative with respect to the argument φ, we suggest the introduction of a second variable η which considers $k\cdot\zeta$ independent of ζ via the following identity:

$$\int \mathrm{d}\eta_a \mathrm{d}\eta_b \delta(\eta_a - k\cdot\zeta_a) \times\int D\eta Dp_\eta \exp\left[\mathrm{i}\int_0^T p_\eta\left(\dot{\eta} - k\cdot\dot{\zeta}\right)\mathrm{d}\tau\right] = 1. \tag{5}$$

Obviously, η is equal to $k\cdot\zeta$ at each time of the evolution and the variables η and p_η are of the same nature as ζ, namely, they are odd (Grassmann) variables.

The Green functions in these two representations are thus simplified to

$$\begin{aligned}\widetilde{G}_{\mathrm{g}}(x,y) &= \exp\left(\gamma\cdot\frac{\partial}{\partial\Gamma}\right)\int_0^\infty \mathrm{d}T\int\frac{\mathrm{d}^4p}{(2\pi)^4}\int\mathrm{d}\varphi_a\mathrm{d}\varphi_b\\ &\times\delta\left(\varphi_a-k\cdot y\right)\int\mathrm{d}\eta_a\mathrm{d}\eta_b\delta(\eta_a-k\cdot\zeta_a)\\ &\times\int D\varphi Dp_\varphi D\eta Dp_\eta\mathcal{D}\zeta\exp\Bigg\{\mathrm{i}p\cdot(x-y)\\ &+\frac{\mathrm{i}T}{2k_0}(p^2-m^2)+\mathrm{i}\int_0^T\Bigg[\frac{g}{k_0}\left(A\cdot p+\frac{g}{2}A^2\right)\\ &+p_\varphi\left(\dot\varphi+\frac{p\cdot k}{k_0}\right)-\frac{2\mathrm{i}g}{k_0}\eta(A'\cdot\zeta)+\mathrm{i}\zeta\cdot\dot\zeta+\\ &p_\eta\left(\dot\eta-k\cdot\dot\zeta\right)\Bigg]\mathrm{d}\tau+\zeta(0)\cdot\zeta(T)\Bigg\}\Bigg|_{\Gamma=0},\end{aligned}\tag{6a}$$

and

$$\begin{aligned}G_{\mathrm{l}}(x,y) &= \exp\left(\gamma\cdot\frac{\partial}{\partial\Gamma}\right)\int_0^\infty\mathrm{d}T\int\frac{\mathrm{d}^4p}{(2\pi)^4}\int\mathrm{d}\chi\\ &\times\exp\left[\frac{-im}{2k_0}\chi\right]\int\mathrm{d}\varphi_a\mathrm{d}\varphi_b\delta\left(\varphi_a-k\cdot y\right)\\ &\times\int\mathrm{d}\eta_a\mathrm{d}\eta_b\delta(\eta_a-k\cdot\zeta_a)\int D\varphi Dp_\varphi D\eta Dp_\eta\mathcal{D}\zeta\\ &\times\exp\Bigg\{\mathrm{i}p\cdot(x-y)+\frac{\mathrm{i}T}{2k_0}(p^2-m^2)\\ &+\mathrm{i}\int_0^T\Bigg[\frac{g}{k_0}\left(A\cdot p+\frac{g}{2}A^2\right)+p_\varphi\left(\dot\varphi+\frac{p\cdot k}{k_0}\right)\\ &-\frac{1}{k_0T}\left[(p+gA)\cdot\zeta+\eta p_\varphi\right]\chi-\frac{2\mathrm{i}g}{k_0}\eta\left(A'\cdot\zeta\right)\\ &+p_\eta\left(\dot\eta-k\cdot\dot\zeta\right)+\mathrm{i}\zeta\cdot\dot\zeta\Bigg]\mathrm{d}\tau\\ &+\zeta\left(0\right)\zeta(T)\Bigg\}\Bigg|_{\Gamma=0}.\end{aligned}\tag{6b}$$

Knowing that in the two preceding expressions of the propagator, the spin variables are subjected to boundary condition reflecting the antiperiodic character of the nature of spin and that the exponential contains an additional term $\zeta\left(0\right)\cdot\zeta(T)$, it is convenient to elude these difficulties by introducing the variable change $\zeta(\tau)\to\omega\left(\tau\right)$

$$\begin{aligned}\omega_\mu\left(\tau\right)&=\dot\zeta_\mu\left(\tau\right),\\ \zeta^\mu(\tau)&=\frac{1}{2}\int_0^T\varepsilon\left(\tau-\tau'\right)\omega^\mu\left(\tau'\right)\mathrm{d}\tau'+\frac{\Gamma^\mu}{2},\\ \varepsilon\left(\tau\right)&=\text{sign of}\tau,\end{aligned}\tag{7}$$

where the velocity $\omega\left(\tau\right)$ keeps the same nature as $\zeta(\tau)$.

We should note that during the initial and final time, the so-called antiperiodic boundary condition is always satisfied: $\zeta^\mu(0)+\zeta^\mu(T)=\Gamma^\mu$. In other words, the velocity variables are not subject to any restrictions, in contrast to the ζ^μ. Besides, following this transformation, a quadratic term in $\omega\left(\tau\right)$ has appeared in the action.

Therefore, the Green functions in the two respective projections become

$$\begin{aligned}\widetilde{G}_{\mathrm{g}}(x,y) &= \exp\left(\gamma\cdot\frac{\partial}{\partial\Gamma}\right)\int_0^\infty\mathrm{d}T\int\frac{\mathrm{d}^4p}{(2\pi)^4}\int\mathrm{d}\varphi_a\mathrm{d}\varphi_b\\ &\times\delta\left(\varphi_a-k\cdot y\right)\int\mathrm{d}\eta_a\mathrm{d}\eta_b\int D\varphi Dp_\varphi D\eta Dp_\eta\mathcal{D}\omega\\ &\times\left(\sqrt{\det\varepsilon}\right)^{-1}\delta\left(\eta_a+\frac{k}{2}\cdot(\omega-\Gamma)\right)\\ &\times\exp\Bigg\{\mathrm{i}p\cdot(x-y)+\frac{\mathrm{i}T}{2k_0}(p^2-m^2)\\ &+\mathrm{i}\int_0^T\Bigg[\frac{g}{k_0}\left(A\cdot p+\frac{g}{2}A^2\right)+p_\varphi\left(\dot\varphi+\frac{p\cdot k}{k_0}\right)\\ &-\frac{\mathrm{i}g}{k_0}\eta A'\cdot(\varepsilon\omega+\Gamma)+p_\eta\left(\dot\eta-k\cdot\omega\right)\\ &-\frac{\mathrm{i}}{2}\omega\cdot\varepsilon\omega\Bigg]\mathrm{d}\tau\Bigg\}\Bigg|_{\Gamma=0},\end{aligned}\tag{8a}$$

and

$$\begin{aligned}G_{\mathrm{l}}(x,y) &= \exp\left(\gamma\cdot\frac{\partial}{\partial\Gamma}\right)\int_0^\infty\mathrm{d}T\int\frac{\mathrm{d}^4p}{(2\pi)^4}\int\mathrm{d}\chi\\ &\times\exp\left[\frac{-im}{2k_0}\chi\right]\int\mathrm{d}\varphi_a\mathrm{d}\varphi_b\delta\left(\varphi_a-k\cdot y\right)\\ &\times\int\mathrm{d}\eta_a\mathrm{d}\eta_b\int D\varphi Dp_\varphi D\eta Dp_\eta\mathcal{D}\omega\left(\sqrt{\det\varepsilon}\right)^{-1}\\ &\times\delta\left(\eta_a+\frac{k}{2}\cdot(\omega-\Gamma)\right)\exp\Bigg\{\mathrm{i}p\cdot(x-y)\\ &+\frac{\mathrm{i}T}{2k_0}(p^2-m^2)+\mathrm{i}\int_0^T\Bigg[\frac{g}{k_0}\left(A\cdot p+\frac{g}{2}A^2\right)\\ &+p_\varphi\left(\dot\varphi+\frac{p\cdot k}{k_0}\right)+p_\eta\left(\dot\eta-k\cdot\omega\right)-\\ &\frac{\mathrm{i}g}{k_0}\eta A'\cdot(\varepsilon\omega+\Gamma)-\frac{\mathrm{i}}{2}\omega\cdot\varepsilon\omega\\ &-\frac{1}{k_0T}\Bigg[\frac{1}{2}\left(p+gA\right)\cdot(\varepsilon\omega+\Gamma)+\\ &\eta p_\varphi\Bigg]\chi\Bigg]d\tau\Bigg\}\Bigg|_{\Gamma=0},\end{aligned}\tag{8b}$$

where we have used the following convolution notation:

$$s\epsilon g\equiv\int_0^T s(\tau)\epsilon\left(\tau-\tau'\right)g(\tau')\mathrm{d}\tau\mathrm{d}\tau'.$$

In order to involve the classical equation of motion in the evolution let us make the following shift

$$\omega^{\mu}(\tau) \longrightarrow \omega^{\mu}(\tau) - \mathrm{i}k^{\mu} \int p_{\eta}(\tau')\varepsilon^{-1}(\tau-\tau')\mathrm{d}\tau'. \tag{9}$$

The functions $\widetilde{G}_{\mathrm{g}}$ and G_{l} therefore become

$$\begin{aligned}\widetilde{G}_{\mathrm{g}}(x,y) = &\exp\left(\gamma\cdot\frac{\partial}{\partial\Gamma}\right)\int_0^{\infty}\mathrm{d}T\int\frac{\mathrm{d}^4p}{(2\pi)^4}\int\mathrm{d}\varphi_a\mathrm{d}\varphi_b \\ &\times\delta\left(\varphi_a - k\cdot y\right)\int\mathrm{d}\eta_a\mathrm{d}\eta_b\int D\varphi Dp_{\varphi}D\eta Dp_{\eta}\mathcal{D}\omega \\ &\times\left(\sqrt{\det\varepsilon}\right)^{-1}\delta\left(\eta_a + \frac{k}{2}\cdot(\omega-\Gamma)\right) \\ &\times\exp\left\{\mathrm{i}p\cdot(x-y) + \frac{\mathrm{i}T}{2k_0}(p^2-m^2)\right. \\ &+\mathrm{i}\int_0^T\left[\frac{g}{k_0}\left(A\cdot p + \frac{g}{2}A^2\right) + p_{\varphi}\left(\dot{\varphi} + \frac{p\cdot k}{k_0}\right)\right. \\ &\left.\left.\left. - \frac{\mathrm{i}g}{k_0}\eta A'\cdot(\varepsilon\omega+\Gamma) + p_{\eta}\dot{\eta} - \frac{\mathrm{i}}{2}\omega\cdot\varepsilon\omega\right]\right\}\right|_{\Gamma=0},\end{aligned} \tag{10a}$$

$$\begin{aligned}G_{\mathrm{l}}(x,y) = &\exp\left(\gamma\cdot\frac{\partial}{\partial\Gamma}\right)\int_0^{\infty}\mathrm{d}T\int\frac{\mathrm{d}^4p}{(2\pi)^4}\int\mathrm{d}\chi \\ &\times\exp\left[\frac{-\mathrm{i}m}{2k_0}\chi\right]\int\mathrm{d}\varphi_a\mathrm{d}\varphi_b\delta\left(\varphi_a - k\cdot y\right) \\ &\times\int\mathrm{d}\eta_a\mathrm{d}\eta_b\int D\varphi Dp_{\varphi}D\eta Dp_{\eta}\mathcal{D}\omega\left(\sqrt{\det\varepsilon}\right)^{-1} \\ &\times\delta\left(\eta_a + \frac{k}{2}\cdot(\omega-\Gamma)\right)\exp\left\{\mathrm{i}p\cdot(x-y)\right. \\ &+\frac{\mathrm{i}T}{2k_0}(p^2-m^2) + \mathrm{i}\int_0^T\left[\frac{g}{k_0}\left(A\cdot p + \frac{g}{2}A^2\right)\right. \\ &+p_{\varphi}\left(\dot{\varphi} + \frac{p\cdot k}{k_0}\right) - \frac{\mathrm{i}g}{k_0}\eta A'\cdot(\varepsilon\omega+\Gamma) \\ &+p_{\eta}\left(\dot{\eta} + \frac{\mathrm{i}p\cdot k}{2Tk_0}\chi\right) - \frac{\mathrm{i}}{2}\omega\cdot\varepsilon\omega - \frac{1}{k_0T} \\ &\left.\left.\left.\times\left[\frac{1}{2}(p+gA)\cdot(\varepsilon\omega+\Gamma) + \eta p_{\varphi}\right]\chi\right]\mathrm{d}\tau\right\}\right|_{\Gamma=0}.\end{aligned} \tag{10b}$$

Consequently, the integration over p_{η} will be straightforward and there appear respectively in the two representations Dirac functions, namely

$$\int Dp_{\eta}\exp\left(\mathrm{i}\int_0^T p_{\eta}\dot{\eta}\mathrm{d}\tau\right) = \delta\left(\dot{\eta}\right), \tag{11a}$$

$$\int Dp_{\eta}\exp\left(\mathrm{i}\int_0^T p_{\eta}\left(\dot{\eta} + \frac{\mathrm{i}p\cdot k}{2Tk_0}\chi\right)\mathrm{d}\tau\right) = \delta\left(\dot{\eta} + \frac{\mathrm{i}p\cdot k}{2Tk_0}\chi\right). \tag{11b}$$

It is remarkable that the arguments of the delta functional are nothing but the classical equations of motion. In fact, it is easy to see that the classical equations of motion related to the (Grassmann) variable ζ obtained from the Lagrangian and multiplied by k_{μ} become simply

$$-2\mathrm{i}k\cdot\dot{\zeta} = 0, \tag{12a}$$

$$-2\mathrm{i}k\cdot\dot{\zeta} + \frac{p\cdot k}{Tk_0}\chi = 0. \tag{12b}$$

That is to say, we again obtain the same argument of the delta functions:

$$\dot{\eta} = 0, \tag{13a}$$

$$\dot{\eta} = -\frac{\mathrm{i}p\cdot k}{2Tk_0}\chi. \tag{13b}$$

The respective solutions to the equations of motion are elementary, and they are given, respectively, by

$$\eta\left(\tau\right) = \eta_a = \mathrm{cons}, \tag{14a}$$

$$\eta(\tau) = \eta_a - \frac{\mathrm{i}p\cdot k}{2Tk_0}\chi\tau. \tag{14b}$$

Let us go back to the Dirac functions or rather to its integral form

$$\delta\left(\eta_a + \frac{k}{2}\cdot(\omega-\Gamma)\right) = \int\mathrm{d}p_{\eta_a}\exp\left[\mathrm{i}p_{\eta_a}\left(\eta_a + \frac{k}{2}\cdot(\omega-\Gamma)\right)\right], \tag{15}$$

where p_{η_a} is an odd (Grassmann) variable and let us include it in the expressions of the $\widetilde{G}_{\mathrm{g}}$ and G_{l}. As the integral over the $\omega^{\mu}(\tau)$ variable has a known standard form,

$$\int\mathcal{D}\omega\exp\left\{\frac{1}{2}\int_0^T\int_0^T\omega_{\mu}(\tau)\varepsilon(\tau-\tau')\omega^{\mu}(\tau')\mathrm{d}\tau\mathrm{d}\tau' + \int_0^T\mathcal{J}_{\mu}(\tau)\omega^{\mu}(\tau)\mathrm{d}\tau\right\},$$

the result will simply become

$$\sqrt{\det\varepsilon}\exp\left\{\frac{1}{2}\int_0^T\int_0^T\mathcal{J}_{\mu}(\tau)\varepsilon^{-1}(\tau-\tau')\mathcal{J}^{\mu}(\tau')\mathrm{d}\tau\mathrm{d}\tau'\right\}, \tag{16}$$

where $\mathcal{J}_{\mu}(\tau)$ is a Grassmannian current, and in our case we have the following respective expressions:

$$\mathcal{J}_{\mu}(\tau) = \frac{g}{2}\eta_a\int A'_{\mu}\left[\varphi(\tau')\right]\varepsilon(\tau'-\tau)\mathrm{d}\tau' + \frac{\mathrm{i}}{2}k_{\mu}p_{\eta_a}, \tag{17a}$$

$$\begin{aligned}\mathcal{J}_{\mu}(\tau) = &\frac{g}{2}\int_0^T\eta(\tau')A'_{\mu}\left[\varphi(\tau')\right]\varepsilon(\tau'-\tau)\mathrm{d}\tau' + \frac{\mathrm{i}}{2}k_{\mu}p_{\eta_a} \\ &+\frac{\mathrm{i}}{2Tk_0}\chi\int_0^T\left(p_{\mu} + gA_{\mu}\left[\varphi(\tau')\right]\right)\varepsilon(\tau'-\tau)\mathrm{d}\tau'.\end{aligned}$$

(17b)

So, after some calculations the Green functions become

$$\widetilde{G}_{\mathrm{g}}(x,y)=\exp\left(\gamma\cdot\frac{\partial}{\partial\Gamma}\right)\int_0^\infty \mathrm{d}T\int\frac{\mathrm{d}^4p}{(2\pi)^4}\int\mathrm{d}\varphi_a\mathrm{d}\varphi_b \times\delta\left(\varphi_a-k\cdot y\right)\int\mathrm{d}\eta_a\mathrm{d}\eta_b\delta\left(\eta_b-\eta_a\right)\int D\varphi Dp_\varphi \times\int\mathrm{d}p_{\eta_a}\exp\left\{\mathrm{i}p\cdot(x-y)+\frac{\mathrm{i}T}{2k_0}(p^2-m^2) +\mathrm{i}\int_0^T\left[\frac{g}{k_0}\left(A\cdot p+\frac{g}{2}A^2\right)+p_\varphi\left(\dot\varphi+\frac{p\cdot k}{k_0}\right) -\frac{\mathrm{i}g}{k_0}\eta_aA'\cdot\Gamma+\frac{1}{T}p_{\eta_a}\left(\eta_a-\frac{k\cdot\Gamma}{2}\right)\right]\mathrm{d}\tau\right\}\Bigg|_{\Gamma=0}, \tag{18a}$$

$$G_{\mathrm{l}}(x,y)=\exp\left(\gamma\cdot\frac{\partial}{\partial\Gamma}\right)\int_0^\infty\mathrm{d}T\int\frac{\mathrm{d}^4p}{(2\pi)^4}\int\mathrm{d}\chi \times\exp\left[\frac{-\mathrm{i}m}{2k_0}\chi\right]\int\mathrm{d}\varphi_a\mathrm{d}\varphi_b\delta\left(\varphi_a-k\cdot y\right) \times\int\mathrm{d}\eta_a\mathrm{d}\eta_b\delta\left(\eta_b-\eta_a+\frac{\mathrm{i}p\cdot k}{2k_0}\chi\right)\int D\varphi Dp_\varphi \times\int\mathrm{d}p_{\eta_a}\exp\left\{\mathrm{i}p\cdot(x-y)+\frac{\mathrm{i}T}{2k_0}(p^2-m^2) +\mathrm{i}\int_0^T\left[\frac{g}{k_0}\left(A\cdot p+\frac{g}{2}A^2\right)+p_\varphi\left(\dot\varphi+\frac{p\cdot k}{k_0} -\frac{\eta\chi}{Tk_0}\right)-\frac{\mathrm{i}g}{k_0}\eta A'\cdot\Gamma+\frac{1}{T}p_{\eta_a}\left(\eta_a-\frac{k\cdot\Gamma}{2} -\frac{\mathrm{i}p\cdot k}{4k_0}\chi\right)-\frac{g^2}{2k_0^2}\eta A'\cdot\varepsilon\eta A' -\frac{1}{2k_0T}(p+gA)\cdot\Gamma\chi -\frac{\mathrm{i}g}{Tk_0^2}\chi(p+gA)\cdot\varepsilon\eta A'\right]\mathrm{d}\tau\right\}\Bigg|_{\Gamma=0}, \tag{18b}$$

where η present in (18b) should be replaced by taking into account (14b). Let us now integrate over p_{η_a} to get, respectively,

$$\int\mathrm{d}p_{\eta_a}\exp\left[\mathrm{i}p_{\eta_a}\left(\eta_a-\frac{k\cdot\Gamma}{2}\right)\right]=\delta\left(\eta_a-\frac{k\cdot\Gamma}{2}\right), \tag{19a}$$

$$\int\mathrm{d}p_{\eta_a}\exp\left[\mathrm{i}p_{\eta_a}\left(\eta_a-\frac{k\cdot\Gamma}{2}-\frac{\mathrm{i}p\cdot k}{4k_0}\chi\right)\right] =\delta\left(\eta_a-\frac{k\cdot\Gamma}{2}-\frac{\mathrm{i}p\cdot k}{4k_0}\chi\right). \tag{19b}$$

From this we get

$$\eta_a=\frac{k\cdot\Gamma}{2}, \tag{20a}$$

$$\eta_a=\frac{k\cdot\Gamma}{2}+\frac{\mathrm{i}p\cdot k}{4k_0}\chi. \tag{20b}$$

One can immediately verify that the antiperiodic boundary condition $\eta_a+\eta_b=k\cdot\Gamma$ is satisfied. By substituting the previous results in (18a) and (18b) we have

$$\widetilde{G}_{\mathrm{g}}(x,y)=\exp\left(\gamma\cdot\frac{\partial}{\partial\Gamma}\right)\int_0^\infty\mathrm{d}T\int\frac{\mathrm{d}^4p}{(2\pi)^4}\int\mathrm{d}\varphi_a\mathrm{d}\varphi_b \times\delta\left(\varphi_a-k\cdot y\right)\int D\varphi Dp_\varphi\exp\left\{\mathrm{i}p\cdot(x-y) +\frac{\mathrm{i}T}{2k_0}(p^2-m^2)+\mathrm{i}\int_0^T\left[\frac{g}{k_0}\left(A\cdot p+\frac{g}{2}A^2\right) +p_\varphi\left(\dot\varphi+\frac{p\cdot k}{k_0}\right)-\frac{\mathrm{i}g}{2k_0}k\cdot\Gamma A'\cdot\Gamma\right]\mathrm{d}\tau\right\}\Bigg|_{\Gamma=0}, \tag{21a}$$

$$G_{\mathrm{l}}(x,y)=\exp\left(\gamma\cdot\frac{\partial}{\partial\Gamma}\right)\int_0^\infty\mathrm{d}T\int\frac{\mathrm{d}^4p}{(2\pi)^4}\int\mathrm{d}\chi \times\exp\left[\frac{-\mathrm{i}m}{2k_0}\chi\right]\int\mathrm{d}\varphi_a\mathrm{d}\varphi_b\delta\left(\varphi_a-k\cdot y\right) \times\int D\varphi Dp_\varphi\exp\left\{\mathrm{i}p\cdot(x-y)+\frac{\mathrm{i}T}{2k_0}(p^2-m^2) +\mathrm{i}\int_0^T\left[\frac{g}{k_0}\left(A\cdot p+\frac{g}{2}A^2\right)+p_\varphi\left(\dot\varphi+\frac{p\cdot k}{k_0} -\frac{k\cdot\Gamma}{2Tk_0}\chi\right)-\frac{\mathrm{i}g}{2k_0}\left(k\cdot\Gamma+\frac{\mathrm{i}p\cdot k}{2k_0}\chi-\frac{\mathrm{i}p\cdot k}{Tk_0}\chi\tau\right) \times A'\cdot\Gamma-\frac{1}{2k_0T}(p+gA)\cdot\Gamma\chi- \frac{g}{4Tk_0^2}\chi k\cdot\Gamma(p+gA)\cdot\varepsilon A' +\frac{g^2p\cdot k}{4Tk_0^3}\chi k\cdot\Gamma\tau A'\cdot\varepsilon A'\right]\mathrm{d}\tau\right\}\Bigg|_{\Gamma=0}. \tag{21b}$$

We integrate afterwards over p_φ:

$$\int\mathrm{d}p_\varphi\exp\left[\mathrm{i}\int_0^T p_\varphi\left(\dot\varphi+\frac{p\cdot k}{k_0}\right)\mathrm{d}\tau\right] =\delta\left(\dot\varphi+\frac{p\cdot k}{k_0}\right), \tag{22a}$$

$$\int Dp_\varphi\exp\left(\mathrm{i}\int_0^T p_\varphi\left(\dot\varphi+\frac{p\cdot k}{k_0}-\frac{k\cdot\Gamma}{2Tk_0}\chi\right)\mathrm{d}\tau\right) =\delta\left(\dot\varphi+\frac{p\cdot k}{k_0}-\frac{k\cdot\Gamma}{2Tk_0}\chi\right), \tag{22b}$$

and we obtain respectively the classical equations of motion

$$\dot\varphi=\frac{-p\cdot k}{k_0}, \tag{23a}$$

$$\dot\varphi=\frac{-p\cdot k}{k_0}+\frac{k\cdot\Gamma}{2Tk_0}\chi. \tag{23b}$$

In fact, these can be obtained by deriving a Lagrangian over the x path and projecting it along the vector k. The even (Grassmann) term characterizes the spin contribution.

These relations characterize the link between τ and φ and in consequence allow us to integrate the interaction term.

The Green functions are then reduced, respectively, to

$$\widetilde{G}_{\rm g}(x,y) = \exp\left(\gamma\cdot\frac{\partial}{\partial\Gamma}\right)\int_0^\infty \mathrm{d}T\int\frac{\mathrm{d}^4p}{(2\pi)^4}\int \mathrm{d}\varphi_a \mathrm{d}\varphi_b \times\delta\left(\varphi_a - k\cdot y\right)\delta\left(\varphi_b-\varphi_a+\frac{p\cdot kT}{k_0}\right) \times\exp\left\{\mathrm{i}p\cdot(x-y)+\frac{\mathrm{i}T}{2k_0}(p^2-m^2)- \frac{\mathrm{i}}{p\cdot k}\int_{\varphi_a}^{\varphi_b}\mathrm{d}\varphi\left[gA\cdot p+\frac{g^2}{2}A^2\right] -\frac{g}{2p\cdot k}(k\cdot\Gamma)\left[A_b-A_a\right]\cdot\Gamma\right\}\Bigg|_{\Gamma=0}, \quad (24a)$$

$$G_{\rm l}(x,y) = \exp\left(\gamma\cdot\frac{\partial}{\partial\Gamma}\right)\int_0^\infty \mathrm{d}T\int\frac{\mathrm{d}^4p}{(2\pi)^4}\int \mathrm{d}\chi \times\exp\left[\frac{-\mathrm{i}m}{2k_0}\chi\right]\int \mathrm{d}\varphi_a\mathrm{d}\varphi_b\delta\left(\varphi_a-k\cdot y\right) \times\delta\left(\varphi_b-\varphi_a+\frac{p\cdot kT}{k_0}-\frac{k\cdot\Gamma}{2k_0}\chi\right) \times\exp\left\{\mathrm{i}p\cdot(x-y)+\frac{\mathrm{i}T}{2k_0}(p^2-m^2)-\frac{\mathrm{i}}{p\cdot k} \times\int_{\varphi_a}^{\varphi_b}\mathrm{d}\varphi\left[gA\cdot p+\frac{g^2}{2}A^2\right]-\frac{g}{2p\cdot k}k \cdot\Gamma\left(A_b-A_a\right)\cdot\Gamma+\frac{\mathrm{i}}{2k_0}\chi\, p\cdot\Gamma+ \frac{\mathrm{i}g}{4k_0}\chi\left(A_a+A_b\right)\cdot\Gamma- \frac{\mathrm{i}g^2}{4k_0 p\cdot k}\chi k\cdot\Gamma A_a\cdot A_b -\frac{\mathrm{i}g}{4k_0p\cdot k}\chi k\cdot\Gamma\left(A_a+A_b\right)\cdot p\right\}\Bigg|_{\Gamma=0}. \quad (24b)$$

Now, using the following integral representation of the Dirac function:

$$\delta\left(\varphi_b-\varphi_a+\frac{p\cdot kT}{k_0}\right) = \int \mathrm{d}p_{\varphi_b}\exp\left[\mathrm{i}p_{\varphi_b}\left(\varphi_b-\varphi_a+\frac{p\cdot kT}{k_0}\right)\right], \quad (25a)$$

$$\delta\left(\varphi_b-\varphi_a+\frac{p\cdot kT}{k_0}-\frac{k\cdot\Gamma}{2Tk_0}\chi\right) \quad (25b)$$
$$= \int \mathrm{d}p_{\varphi_b}\exp\left[\mathrm{i}p_{\varphi_b}\left(\varphi_b-\varphi_a+\frac{p\cdot kT}{k_0}-\frac{k\cdot\Gamma}{2k_0}\chi\right)\right],$$

we make the following shift: $p\to p-kp_{\varphi_b}$ and then by integrating over the fermionic time, the propagator in the local projection will be rewritten

$$G_{\rm l}(x,y) = \frac{-\mathrm{i}}{2k_0}\exp\left(\gamma\cdot\frac{\partial}{\partial\Gamma}\right)\int_0^\infty \mathrm{d}T\int\frac{\mathrm{d}^4p}{(2\pi)^4} \times\left[\left(-p\cdot\Gamma+m\right)\left(1-\frac{g}{2p\cdot k}k\cdot\Gamma\left(A_b-A_a\right)\cdot\Gamma\right) -\frac{g}{2}\left(A_a+A_b\right)\cdot\Gamma+\frac{g}{2p\cdot k}k\cdot\Gamma\left(A_a+A_b\right)\cdot p +\frac{g^2}{2p\cdot k}k\cdot\Gamma A_a\cdot A_b-\frac{g^2}{2p\cdot k}k\cdot\Gamma A_a\cdot\Gamma A_b\cdot\Gamma\right] \times\exp\left\{\mathrm{i}p\cdot(x-y)+\frac{\mathrm{i}T}{2k_0}(p^2-m^2) -\frac{\mathrm{i}}{p\cdot k}\int_{k\cdot y}^{k\cdot x}\left[gA\cdot p+\frac{g^2}{2}A^2\right]\mathrm{d}\varphi\right\}\Bigg|_{\Gamma=0}. \quad (26)$$

Finally, let us proceed to the derivation related to the variables Γ. After some straightforward computations the propagators $\widetilde{G}_{\rm g}$ and $G_{\rm l}$ are respectively rewritten as

$$\widetilde{G}_{\rm g}(x,y) = \int_0^\infty \mathrm{d}T\int\frac{\mathrm{d}^4p}{(2\pi)^4}\left[1-\frac{g}{2p\cdot k}\widehat{k}\left(\widehat{A}_b-\widehat{A}_a\right)\right] \times\exp\left\{-\mathrm{i}p\cdot(x-y)+\frac{\mathrm{i}T}{2k_0}(p^2-m^2) -\frac{\mathrm{i}}{p\cdot k}\int_{k\cdot y}^{k\cdot x}\left[gA\cdot p-\frac{g^2}{2}A^2\right]\mathrm{d}\varphi\right\}, \quad (27a)$$

$$G_{\rm l}\left(x,y\right) = \frac{\mathrm{i}}{2k_0}\int_0^\infty \mathrm{d}T\int\frac{\mathrm{d}^4p}{(2\pi)^4}\left\{\left(\widehat{p}+m\right)\left[1+\frac{g}{2p\cdot k}\widehat{k} \times\left(\widehat{A}_b-\widehat{A}_a\right)\right]-g\widehat{A}_b+\frac{g^2}{2p\cdot k}\widehat{k}\widehat{A}_a\widehat{A}_b -\frac{g^2}{p\cdot k}\widehat{k}A_a\cdot A_b+\frac{g}{p\cdot k}\widehat{k}A_b\cdot p\right\} \times\exp\left\{-\mathrm{i}p\cdot(x-y)+\frac{\mathrm{i}T}{2k_0}(p^2-m^2) -\frac{\mathrm{i}}{p\cdot k}\int_{k\cdot y}^{k\cdot x}\mathrm{d}\varphi\left[gA\cdot p-\frac{g^2}{2}A^2\right]\right\}. \quad (27b)$$

Note that the derivation in (24a) and (26) involves an antisymmetrization, for example,

$$\exp\left(\gamma^\mu\cdot\frac{\partial}{\partial\Gamma^\mu}\right)\Gamma_\mu\Gamma_\nu\Big|_{\Gamma=0} = \frac{1}{2}\left(-\gamma_\nu\gamma_\mu+\gamma_\mu\gamma_\nu\right),$$

and the change variable $p\to -p$ has also been made.

According to (1a), the dynamics of the system is thus totally determined by the following expression in the global projection:

$$G_{\rm g}(x,y) = \frac{\mathrm{i}}{2k_0}\int_0^\infty \mathrm{d}T\int\frac{\mathrm{d}^4p}{(2\pi)^4}\left\{\left(\widehat{p}+m\right)\left[1+\frac{g}{2p\cdot k}\widehat{k}\right.\right.$$

$$\times\left(\widehat{A}_b-\widehat{A}_a\right)\Bigg]-g\widehat{A}_b+\frac{g^2}{2p\cdot k}\widehat{k}\widehat{A}_a\widehat{A}_b$$
$$-\frac{g^2}{p\cdot k}\widehat{k}A_a\cdot A_b+\frac{g}{p\cdot k}\widehat{k}A_b\cdot p\Bigg\}$$
$$\times\exp\Bigg\{-\mathrm{i}p\cdot(x-y)+\frac{\mathrm{i}T}{2k_0}(p^2-m^2)$$
$$-\frac{\mathrm{i}}{p\cdot k}\int_{k\cdot y}^{k\cdot x}\mathrm{d}\varphi\left[gA\cdot p-\frac{g^2}{2}A^2\right]\Bigg\}. \tag{28}$$

We notice that the two representations give the same result, $G_\mathrm{l}=G_\mathrm{g}$.

After symmetrization the propagator related to the Dirac particle interacting with a plane wave field is finally

$$G_{\mathrm{l,g}}(x,y)=\frac{\mathrm{i}}{2k_0}\int_0^\infty\mathrm{d}T\int\frac{\mathrm{d}^4p}{(2\pi)^4}$$
$$\times\left[1+\frac{g}{2p\cdot k}\widehat{k}\widehat{A}_b\right](\widehat{p}+m)\left[1-\frac{g}{2p\cdot k}\widehat{k}\widehat{A}_a\right]$$
$$\times\exp\Bigg\{-\mathrm{i}p\cdot(x-y)+\frac{\mathrm{i}T}{2k_0}(p^2-m^2)$$
$$-\frac{\mathrm{i}}{p\cdot k}\int_{k\cdot y}^{k\cdot x}\mathrm{d}\varphi\left[gA\cdot p-\frac{g^2}{2}A^2\right]\Bigg\}. \tag{29}$$

This result agrees with that of [3].

2 Conclusion

In conclusion, we have calculated the exact Green function related to a Dirac particle interacting with a plane wave field using the formalism of global and local projection proposed recently by Alexandrou et al. Besides, we have shown, thanks to the incorporation of two simple identities, the importance of classical paths (projected in the direction of the plane wave) in determining the Green function in the two cases, global and local.

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